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Comparison of Conventional and Fractal Phased Arrays

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ABSTRACT

We consider an unweighted conventional phased array and compare its performance characteristics such as the array factor, half-power beamwidth, directivity, number of elements and side-lobe levels to a fractal array equivalent. We show that the Cantor fractal set can be used to thin the array structure in such a way as to consist of active radiating elements and inactive elements the latter of which can be used for other functions. We demonstrate that the conventional array 'defocuses' its energy for frequencies other than its design frequency and behaves like a point source. On the other hand the fractal array maintains its beamforming capability for various frequencies which signifies that it has a multiband response. The fractal array is shown to be superior to the conventional array except when it comes to the side-lobe level where the conventional array results are better.

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Executive Summary

We consider an unweighted conventional phased array and compare its performance characteristics such as the array factor, half-power beamwidth, directivity, number of elements and side-lobe levels to a fractal array equivalent. We show that the Cantor fractal set can be used to thin the array structure in such a way as to consist of active radiating elements and inactive elements the latter of which can be used for other functions. We demonstrate that the conventional array 'defocuses' its energy for frequencies other than its design frequency and behaves like a point source. On the other hand the fractal array maintains its beamforming capability for various frequencies which signifies that it has a multiband response. The fractal array is shown to be superior to the conventional array except when it comes to the side-lobe level where the conventional array results are better.

The theoretical results presented here can be expanded to planar and other types of arrays. Most importantly, the approach can be readily verified experimentally by making use of the Microwave Radar Branch XPAR II research facilities.

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Dr Aris Alexopoulos is with the Microwave Radar Branch in the Electronic Warfare and Radar Division. His research interests involve such areas as electromagnetic theory, metamaterial and advanced device research and phased array analysis amongst other things.

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1 Introduction

A fractal is a recursively generated object that has fractional dimension [1]-[3]. The study of objects with fractal geometry in combination with electromagnetic theory is known as fractal electrodynamics. Fractal electrodynamics is used in the investigation of a new class of radiation, propagation and scattering problems. One of the most interesting areas of research is concerned with antenna theory and design. The idea is to investigate the possibility of improved antenna performance based on fractal objects as opposed to the more traditional use of Euclidean geometries. Fractal geometries can be used in the study of the radiation pattern of independent antennas or to examine the performance of an ensemble of elements in a phased array. Kim and Jaggard [4] were one of the first to report on a way to design low sidelobe arrays which was based on random fractals. Various fractal designs have been used in the pursuit of antennas or arrays with high directivity with varying success [5]-[7]. One promising performance characteristic of fractal arrays is the ability to form a beam pattern for varying frequencies, ie, to have multiband response. In this report we will examine a particular class of fractals known as the Cantor set. It will be shown that the array factor $f(\theta)$ for varying frequencies for such an array still focuses the radiating energy while a conventional linear array does not. Comparison is also made of the directivity ($D(\theta)$) half-power beamwidth (θ_H) and side-lobe level (SLL) between the two array types. In both cases the array is isotropic which means that window functions are unity for all radiating elements.

2 Conventional Linear Phased Arrays

Linear phased arrays have been analysed in some detail by many including the author [8]-[9] and for this reason we will not dwell on the details here but only highlight the essential results. A linear array generally has N equally spaced elements with uniform excitations and symmetrical distributions for the magnitude amplitudes in the array, as well as a constant inter-element phase shift α of radiating electromagnetic waves at angle θ from the plane of the array-refer to Fig. 1. For an even and odd number of elements we can express the array factor $f(\psi)$ as,

$$f(\psi) = 2\sum_{m=1}^{N} a_m \cos\left[\left(m - \tfrac{1}{2}\right)\psi\right] \tag{1}$$

when N is an even number of radiating elements ($N = 2, 4, 6, ...$) and

$$f(\psi) = a_0 + 2\sum_{m=1}^{N} a_m \cos\left[m\psi\right] \tag{2}$$

when N is an odd number of radiating elements ($N = 1, 3, 5, ...$). Here $\psi \equiv \psi(\theta) = \beta d \cos(\theta) + \alpha$, $\beta = 2\pi/\lambda$; λ is the wavelength, d is the element spacing, α is the phase factor, θ is the angle measured from the line of the array, a_m is the magnitude of the amplitude for the m^{th} element on either side of the array midpoint and a_0 denotes the amplitude of the centre element when N is odd. The coefficients a_m are the windows of the array factor and there are many approaches that can be used for their derivation. However, in this report we will only be concerned with unweighted array factors whereby

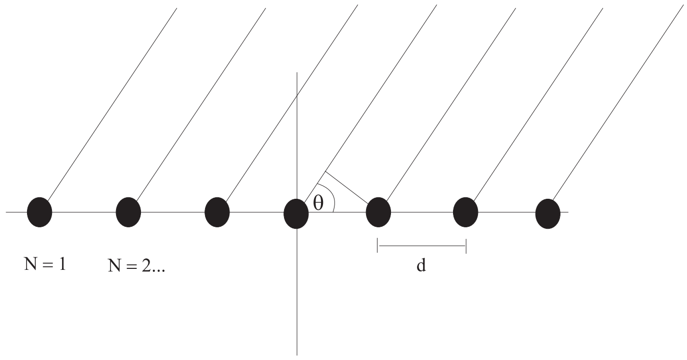


Figure 1: *Linear array with elements radiating at angle θ and phase-shift α. The inter-element separation is d*

all $a_m = 1$ in (1) and (2). This is done in order to facilitate a direct comparison with the array factors of the fractal analogue as well as being simpler for experimental verification of the theory presented here.

3 Fractal Linear Phased Arrays

Fractal array radiation patterns have been studied using generating algorithms that recursively replicate a fundamental subarray structure or pattern of elements that are then used in the beamforming process with the fractal dimension of such arrays being less than one, $d < 1$. By considering an $n = 1$ fractal pattern to be the fundamental subarray to be replicated and generated successively, higher orders in the generating function ($n > 1$) can be used to construct an overall array with radiating elements either turned on or off in a fractal manner that in turn determines the desired performance of the array. This approach belongs to a special class of thinning arrays. In general, the array factor $f(\psi(\theta))$ can be expressed in the form:

$$f(\delta, \psi(\theta)) = \prod_{n=1}^{N} g(\delta, \psi(\theta)) \tag{3}$$

where $g(.)$ represents the array factor of the fundamental subarray elements that are to be expanded or iterated and $\psi(\theta)$ is defined above. In (3), n corresponds to the number of iterations while δ is the scaling parameter or expansion factor that determines how large the array becomes for each successive application of the generating fundamental subarray. To investigate how a conventional linear array as represented by (1) and (2) compares with its fractal equivalent, it is necessary to obtain a generating function $g(\delta, \psi(\theta))$ that self replicates in a linear fashion (linear array) and where the fundamental subarray structure comprising of radiating elements is obtained from (1) and (2). One fractal generator that can be used in this instance is that of the Cantor fractal set - see Fig. 2. We make the connection between this fractal behaviour and say (2) for a conventional array by noting that we can choose appropriate elements to be turned on or off (or removed) from the linear array. Hence if we chose $N = 1$, $a_0 = 0$ and $a_1 = 1$ in (2) we find that what remains is two radiating elements represented by ‘1’ and a switched off element as represented by

Figure 2: *The Cantor fractal set is shown for the first $n = 4$ iterations. Each solid line corresponds to a radiating element while the gaps in between the lines correspond to switched off elements. The iterations here are in direct analogue to the results shown in Table 1*

'0', ie, the fundamental subarray consists of 101 elements and from (2) the generating array factor for this fundamental subarray is

$$g(\delta, \psi(\theta)) = 2\cos(\psi(\theta)) \tag{4}$$

This expression represents the first iteration of the subarray and is the $n = 1$ term in the Cantor fractal set. From the pattern multiplication theorem we find that higher orders in n can be obtained by products of the general term

$$g(\delta, \psi(\theta)) = 2\cos(\delta^{n-1}\psi(\theta)) \tag{5}$$

Given that the fundamental generating subarray has been chosen here to be of the form '101' we select an expansion factor of $\delta = 3$ for the Cantor set and by using (3) we then have the fractal array factor[1]

$$f(3, \psi(\theta)) = 2\prod_{n=1}^{N}\cos(3^{n-1}\psi(\theta)) \tag{6}$$

Equation (6) now represents the array factor for the nth generation or iteration in such a way that each active element '1' is replaced by '101' and each inactive element '0' is replaced by '000' for each subsequent iteration. Hence for $n = 1$ we have '101' and for $n = 2$ we obtain '101000101' and so on as shown in Table 1. Notice that this behaviour correlates with the Cantor fractal set as can be shown in Fig. 2. Suppose that in designing the fractal or conventional array we chose a specific operating frequency f_0 for the array. Generally at this frequency we obtain the best performance from the array depending on what specifications we are interested in. A conventional linear array is therefore expected to focus most of its energy in a beam rather than radiate the field uniformly in all directions. On the other hand a fractal array is known to exhibit multiband characteristics for frequencies other than just the design frequency f_0:

$$f_n = f_0\delta^{-n} \equiv 3^{-n}f_0 \tag{7}$$

[1] Note that we can use a different subarray set to carry out this analysis instead of the '101' radiating elements as used here for example.

Table 1: *The element distribution for the fractal linear array*

n	element pattern	active elements	total elements
1	101	2	3
2	101000101	4	9
3	101000101000000000101000101	8	27
4	101000101000000000101000101 000000000000000000000000000 101000101000000000101000101	16	81

for each iteration of order $n = 0, 1, 2, 3, ..., N-1$. Another parameter of interest for comparing the fractal and conventional linear array is the directivity $D(\theta)$ that is obtained from

$$D(\theta) = 2\frac{f^2(\pi/2u)}{\int_{-1}^{1} f^2(\pi/2u)du} \tag{8}$$

where we define $\psi = \pi/2u$ and $u = \cos(\theta)$ and where,

$$f^2(\pi/2u) = \prod_{n=1}^{N} \cos\left(\frac{3^{n-1}}{2}\pi u\right) \tag{9}$$

Substituting (9) into (8) we finally obtain the directivity for a linear Cantor fractal array as a function of the angle θ:

$$D(\theta) = 2^N \prod_{n=1}^{N} \cos^2\left(\frac{3^{n-1}}{2}\pi u\right) \tag{10}$$

Evidently the maximum directivity D_{max} occurs when $u = \cos(\pi/2) = 0$ hence we have $D_{max} = D(0)$,

$$D(0) = 2^N \tag{11}$$

or $D_{dB}(0) = 3.01N$ in dB where $N = 1, 2, 3, ...$ Finally it is worth noting that the nulls of the fractal array can easily be calculated from the fact that

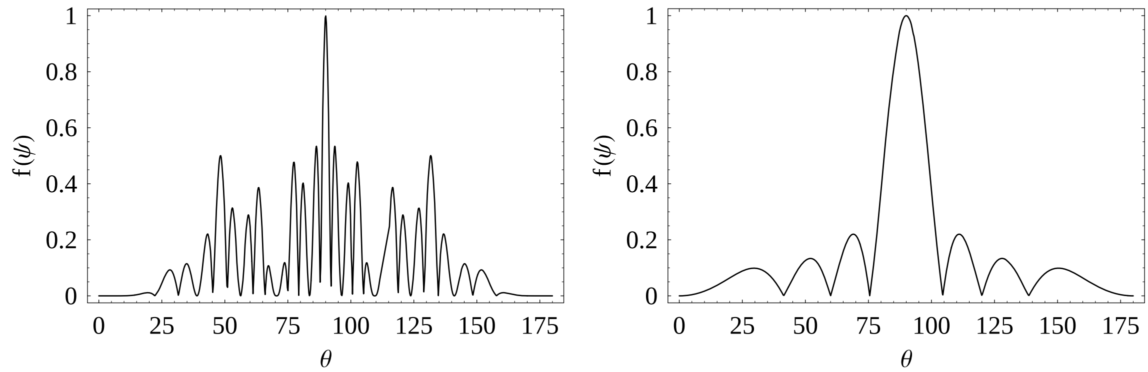
$$\cos\left(\frac{3^{n-1}}{2}\pi u\right) = 0 \tag{12}$$

with solutions given by

$$u_j^{(N)} = \pm(2j-1)(1/3)^{N-1} \tag{13}$$

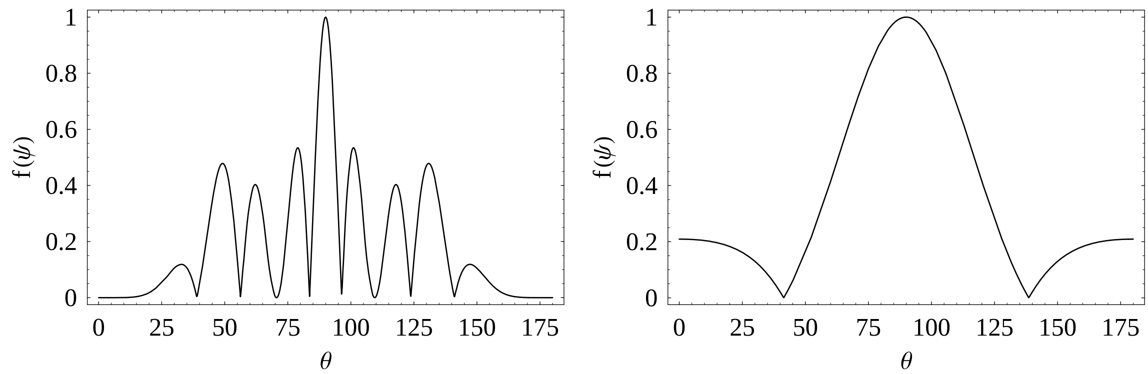
Thus the angles at which the nulls appear are now determined from

$$\theta_j^{(N)} = \cos^{-1}\left(\pm(2j-1)(1/3)^{N-1}\right) \tag{14}$$



(i) Fractal array (ii) Conventional array

Figure 3: *Plots showing the radiation response of fractal and conventional phased arrays for a design frequency of f_0=8.1 GHz*



(i) Fractal array (ii) Conventional array

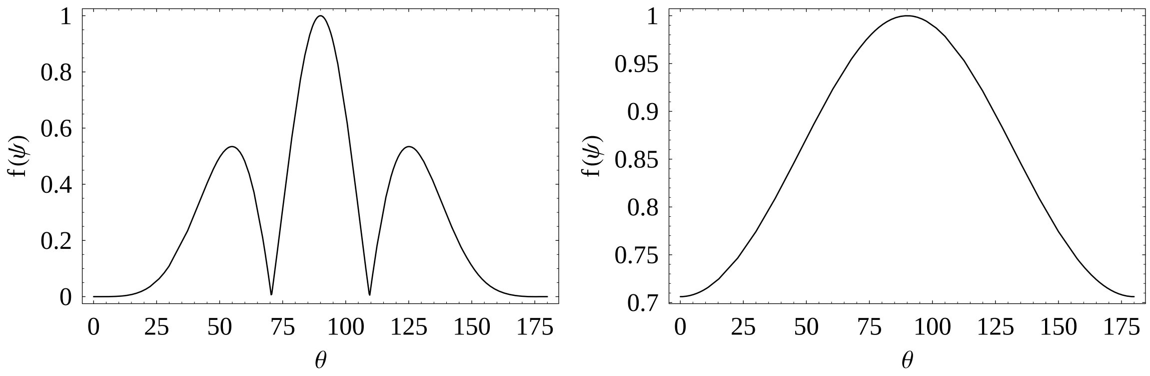
Figure 4: *Plots showing the radiation response of fractal and conventional phased arrays for a frequency of f_1=2.7 GHz*

where $j = 1, 2, 3, ..., 1/2(1+3^{N-1})$ since there are $1+3^{N-1}$ nulls for the array. It is also important to check that such an array is indeed a fractal array. This can be determined by considering the dimension[2] d which for the array considered here must lie in the interval $0 \le d \le 1$,

$$d = \frac{\log(\frac{\delta+1}{2})}{\log(\delta)} = \frac{\log(2)}{\log(3)} \tag{15}$$

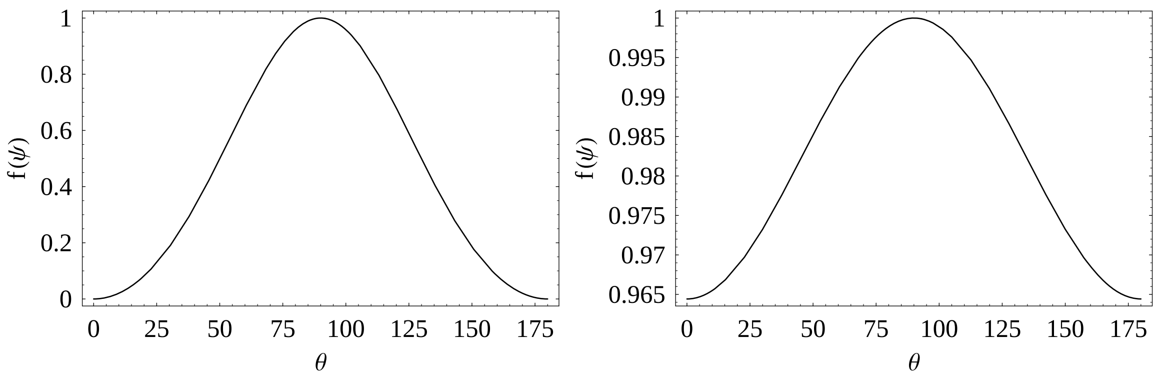
hence the dimension is indeed that of a fractal since $d = 0.6309$. Suppose that we consider a linear array design such that the operating frequency is $f_0 = 8.1$ GHz. Then for this frequency the corresponding wavelength is $\lambda_0 = 0.037$ m. Furthermore let the inter-element spacing be $d = \lambda_0/4$ and the phase shift between the elements be set to zero ($\alpha = 0$) for simplicity. We will compare a 16-element conventional array with its corresponding fractal version. For the latter this means performing $N = 4$ iterations of the fundamental

[2]Not to be confused with the inter-element spacing d.



(i) Fractal array (ii) Conventional array

Figure 5: *Plots showing the radiation response of fractal and conventional phased arrays for a frequency of f_2=0.9 GHz*



(i) Fractal array (ii) Conventional array

Figure 6: *Plots showing the radiation response of fractal and conventional phased arrays for a frequency of f_3=0.3 GHz*

subarray configuration ‘101’ as given in Table 1. Notice that in order to obtain 16 active elements (as in the conventional array case) a total of 81 elements are generated by the fractal iterations in which the majority are switched off with 16 elements left in the radiating mode. Hence, by separating these active left-over elements at multiples of the quarter-wavelength separations between two such elements we can obtain an array factor which can be compared to that of a conventional linear array. Also it is worth noting that while these inactive elements are not needed for the beamforming itself, they can be used for other purposes and this is especially true for the planar version of a fractal array to which these results can be easily extended. Returning to the design frequency of $f_0 = 8.1$ GHz we notice that the fractal array operates at another three frequencies aside from the design frequency, ie, as given by (7):

$$f_0 = 3^0 f_0 = 8.1 \quad GHz \tag{16}$$
$$f_1 = 3^{-1} f_0 = 2.7 \quad GHz \tag{17}$$
$$f_2 = 3^{-2} f_0 = 0.9 \quad GHz \tag{18}$$

Table 2: *Conventional linear phased array*

f (GHz)	D (dB)	$\theta_H(degrees)$	SLL (dB)
8.1	9.12	17.00	-13.14
2.7	4.63	53.50	-13.55
0.9	0.89	-	$-\infty$
0.3	0.10	-	$-\infty$

Table 3: *Fractal linear phased array*

f (GHz)	D (dB)	$\theta_H(degrees)$	SLL (dB)
8.1	12.04	4.00	-5.45
2.7	9.03	12.95	-5.44
0.9	6.02	24.50	-5.44
0.3	3.01	80.45	$-\infty$

$$f_3 = 3^{-3} f_0 = 0.3 \quad GHz \tag{19}$$

The array factor is compared for both conventional and fractal arrays for these different frequencies as shown in Figs 3-6. As can be seen, the fractal version still forms a beam pattern not only for the design frequency f_0 but also for the other frequencies as well showing that such an approach does in fact display multiband behaviour. Conversely, for the conventional array, as the frequency changes from the design frequency it behaves more like a point source and loses its beam focusing ability. Finally, as can be seen in Table 2 and Table 3, the fractal array has greater directivity and smaller half-power beamwidth θ_H compared to the conventional linear array. However the side-lobe levels (SLL) are greater for the fractal array than they are for the conventional array but we are reminded that this can be reversed if appropriate weighting functions are calculated. On the other hand, it may be possible to turn on some of the inactive elements as shown in Table 1, even if only momentarily, so that the side-lobes are reduced for a particular scan mode before reverting back to the original scan configuration.

4 Conclusion

We have presented one possible class of fractals, namely the Cantor set, in the study of an unweighted fractal linear array and have compared it with a conventional unweighted array described by a $sinc(z)$ function. In most performance characteristics the fractal array shows better response compared to a conventional array. The method can be extended to 2D array structures whereby element 'thinning' based on fractal iterations might improve some aspects of performance for such array structures.

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